Math 2058, HW 3. Due: 22 Oct 2024, before 11:59 pm

(1) Suppose (x_n) is a bounded sequence of real numbers. Define
- $L_1 := \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k$;
- $L_2 := \sup\{w \in \mathbb{R} : x_m < w \text{ for at most finitely many } m\}$;
- $L_3 := \inf S$ where S denotes the set of sub-sequential limit of (x_n).

Show that $L_1 = L_2 = L_3$.

(2) Suppose (x_n) is a bounded sequence of positive real numbers, show that

$$\limsup_{n\to+\infty} x_n^{1/n} \leq \limsup_{n\to+\infty} \frac{x_{n+1}}{x_n}.$$

Can we improve the inequality to equality? Justify your answer. Here we define lim sup using $\inf_n \sup_{k\geq n} x_k$.

(3) Show that if (x_n) is a unbounded sequence, then there exists a sub-sequent (x_{n_k}) such that $x_{n_k}^{-1} \to 0$ as $k \to +\infty$.

(4) Suppose every subsequent of (x_n) has a subsequent converging to 0, Show that $x_n \to 0$.

(5) If $x_1 < x_2$ and $x_n = \frac{1}{4}x_{n-1} + \frac{3}{4}x_{n-2}$ for $n > 2$. Show that (x_n) is convergent. Find its limit.

(6) Let $p \in \mathbb{N}$, give an example of sequence (x_n) that is not Cauchy but satisfies $|x_{n+p} - x_n| \to 0$ as $n \to +\infty$.